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# On solutions of the wave equation with homogeneous Cauchy data

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## 1 Introduction

This is based on our recent paper [4].
We consider the Cauchy problem of the free wave equation

$$(FW) \quad \begin{cases} u_{tt} - \Delta u = 0, & (t,x) \in (0,\infty) \times \mathbf{R}^n \equiv \mathbf{R}^{1+n}_+, \\ u|_{t=0} = \phi,\ u_t|_{t=0} = \psi, & x \in \mathbf{R}^n, \end{cases}$$

with the initial data given by homogeneous functions such as

$$\phi(x) = |x|^{-p}, \quad \psi(x) = |x|^{-p-1}. \tag{1.1}$$

These initial data are of special interest in view of the applications for self-similar solutions of wave equations with power type nonlinearity:

$$u_{tt} - \Delta u = |u|^{\alpha}, \quad (t,x) \in \mathbf{R}^{1+n}_+. \tag{1.2}$$

In this paper we study explicit behavior of solutions to $FW$ with special attention on the propagation of singularity.

Precisely, a typical estimate to be shown takes the form

$$|u(t,x)| \le C(t+|x|)^{-\frac{n-1}{2}} \big|t-|x|\big|^{-p+\frac{n-1}{2}}, \quad (t,x) \in \mathbf{R}^{1+n}_+, \tag{1.3}$$

for the initial data such as (1.1), where $p > \frac{n-1}{2}$. This estimate shows that the singularity of the initial data at the origin propagates along the light cone with specific order there. Moreover, we observe that the order of singularity of solutions is less than that of the corresponding initial data for each $t > 0$. Below we also prove the optimality of these estimates. Estimate (1.3) also implies the following integrability

$$u(t,\cdot) \in L^r(\mathbf{R}^n), \quad t > 0 \tag{1.4}$$

for some $r$. This fact has an interest because $u(0,\cdot) = \phi \notin L^r(\mathbf{R}^n)$ for any $r$.

Estimates of the form (1.3), (1.4) for solutions of $FW$ with homogeneous initial data such as (1.1) have been used to construct self-similar solutions to (1.2). We call $u$ a self-similar solution of (1.2) if $u$ satisfies

$$u(t,x) = \lambda^{\frac{2}{\alpha-1}} u(\lambda t, \lambda x), \quad (t,x) \in \mathbf{R}_+^{1+n} \tag{1.5}$$

for all $\lambda > 0$. By the condition (1.5) initial data of self-similar solutions must be homogeneous functions such as (1.1).

In fact, Pecher [6] proved (1.4) when $n = 3$ to construct self-similar solutions satisfying

$$\sup_{t>0} t^{\mu} \|u(t)\|_{L^r} < \infty$$

for suitable $\mu$, $r$. The case $n \geq 2$ is treated by Ribaud-Youssfi [8], where they showed (1.4) for a class of initial data in terms of some homogeneous Besov spaces containing homogeneous functions such as (1.1). Pecher [7] also proved (1.3) when $n = 3$ to construct self-similar solutions satisfying

$$\sup_{|x|\neq t} (t+|x|)\big|t-|x|\big|^{\frac{2}{p-1}-1} |u(t,x)| < \infty.$$

Our estimate (1.3) seems to give a foundation to generalize the last result for higher dimensions.

Our method to obtain the estimate like (1.3) is based on Fourier representation of solutions of $FW$ and some results on oscillatory integrals. More precisely, we divide the representation into high freqency part and low frequency part. As we shall see below, the high frequency part contributes to the formation of singularity along the light cone and the low frequency part to the decay rate as $|x| \to \infty$.

To investigate the behavior of high frequency part of solutions we use the asymptotic expansion of oscillatory integrals over the unit sphere. This consideration has been used in Miyachi [5] to prove the boundedness, together with unboundedness, of some Fourier multipliers associated with the wave equation. Concerning the low frequency part, we also consider an oscillatory integral over the sphere as above and derive its decay estimate via stationary phase method.

# 2 Main Estimates

In this section we give the estimates such as (1.3), (1.4) for solutions of $FW$ with homogeneous functions as initial data. We consider the solution of $FW$ given by

$$u(t) = \cos[(-\Delta)^{\frac{1}{2}}t]\phi + (-\Delta)^{-\frac{1}{2}} \sin[(-\Delta)^{\frac{1}{2}}t]\psi,$$

$$
\begin{aligned}
&\cos[(-\Delta)^{\frac{1}{2}}t]\phi = \mathcal{F}^{-1}\big[\cos t|\xi|\,\widehat{\phi}(\xi)\big],\\
&(-\Delta)^{-\frac{1}{2}}\sin[(-\Delta)^{\frac{1}{2}}t]\psi = \mathcal{F}^{-1}\big[|\xi|^{-1}\sin t|\xi|\,\widehat{\phi}(\xi)\big].
\end{aligned}
$$

Here $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse, respectively and $\widehat{\phi} = \mathcal{F}\phi$. Our main results are as follows.

**Theorem 2.1** *Let $n \geq 2$ and let $0 < p < \frac{n+1}{2}$. We assume that $\phi$, $\psi \in C^\infty(\mathbf{R}^n\backslash\{0\})$ are homogeneous of degree $-p$, $-p-1$, respectively. Then we have for any $t > 0$*

$$
\cos[(-\Delta)^{\frac{1}{2}}t]\phi,\ (-\Delta)^{-\frac{1}{2}}\sin[(-\Delta)^{\frac{1}{2}}t]\psi \in C^\infty(\mathbf{R}^n \setminus \{|x| = t\}).
$$

*Moreover, we have the following estimates:*

(1) *For $\frac{n-1}{2} < p < \frac{n+1}{2}$,*

$$
\begin{aligned}
&|\cos[(-\Delta)^{\frac{1}{2}}t]\phi(x)| \leq C(t+|x|)^{-\frac{n-1}{2}}\big|t-|x|\big|^{-p+\frac{n-1}{2}},\\
&|(-\Delta)^{-\frac{1}{2}}\sin[(-\Delta)^{\frac{1}{2}}t]\psi(x)| \leq Ct(t+|x|)^{-\frac{n+1}{2}}\big|t-|x|\big|^{-p+\frac{n-1}{2}}.
\end{aligned}
$$

(2) *For $p = \frac{n-1}{2}$,*

$$
\begin{aligned}
&|\cos[(-\Delta)^{\frac{1}{2}}t]\phi(x)| \leq C(t+|x|)^{-\frac{n-1}{2}}\big(1+\log^-\big|1-|x/t|\big|\big),\\
&|(-\Delta)^{-\frac{1}{2}}\sin[(-\Delta)^{\frac{1}{2}}t]\psi(x)| \leq Ct(t+|x|)^{-\frac{n+1}{2}}\big(1+\log^-\big|1-|x/t|\big|\big),
\end{aligned}
$$

*where* $\log^- s = \max(-\log s, 0)$.

(3) *For $0 < p < \frac{n-1}{2}$,*

$$
\begin{aligned}
&|\cos[(-\Delta)^{\frac{1}{2}}t]\phi(x)| \leq C(t+|x|)^{-p},\\
&|(-\Delta)^{-\frac{1}{2}}\sin[(-\Delta)^{\frac{1}{2}}t]\psi(x)| \leq Ct(t+|x|)^{-p-1}.
\end{aligned}
$$

**Remark 2.1** *(1) The condition $p < \frac{n+1}{2}$ is related to local integrability in space of solutions.*
*(2) In the case of $0 < p < \frac{n-1}{2}$, we further obtain continuous differentiability and Hölder continuity of the above Fourier multipliers near the light cone. In fact, they belong to the class*

$$
C^{[\frac{n-1}{2}-p]_-,\,\frac{n-1}{2}-p-[\frac{n-1}{2}-p]_--\varepsilon}(\mathbf{R}^n),
$$

*for $\varepsilon > 0$, where $[s]_- = [s]$ if $s \in \mathbf{R}_+ \setminus \mathbf{N}$, $[s]_- = [s]-1$ if $s \in \mathbf{N}$, and $[s]$ is an integer part of $s$. Here $\mathbf{N} = \{1, 2, \cdots\}$.*

**Theorem 2.2** *Let $n \geq 2$ and let $0 < p < \frac{n}{2}$. We assume $\phi$, $\psi \in C^\infty(\mathbf{R}^n \setminus \{0\})$ are homogeneous of degree $-p$, $-p-1$, respectively. Let $t > 0$. Then:*

(1) *For $\frac{n-1}{2} \leq p < \frac{n}{2}$, we have*

$$\cos[(-\Delta)^{\frac{1}{2}}t]\phi \in L^r(\mathbf{R}^n) \quad \text{if} \quad p - \tfrac{n-1}{2} < \tfrac{1}{r} < \tfrac{p}{n},$$
$$(-\Delta)^{-\frac{1}{2}} \sin[(-\Delta)^{\frac{1}{2}}t]\psi \in L^r(\mathbf{R}^n) \quad \text{if} \quad p - \tfrac{n-1}{2} < \tfrac{1}{r} < \tfrac{p+1}{n}.$$

(2) *For $0 < p < \frac{n-1}{2}$, we have*

$$\cos[(-\Delta)^{\frac{1}{2}}t]\phi \in L^r(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n) \quad \text{if} \quad 0 \leq \tfrac{1}{r} < \tfrac{p}{n},$$
$$(-\Delta)^{-\frac{1}{2}} \sin[(-\Delta)^{\frac{1}{2}}t]\psi \in L^r(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n) \quad \text{if} \quad 0 \leq \tfrac{1}{r} < \tfrac{p+1}{n}.$$

**Remark 2.2** *(1) The condition $p < \frac{n}{2}$ is necessary for the existence of $1/r$ over the intervals above.*
*(2) For the conculusion of Theorem 2.2(1) we need not assume the regularity on $\phi$, $\psi$ so much. In fact, we really need*

$$\phi \in C^{n+2}(\mathbf{R}^n \setminus \{0\}), \quad \psi \in C^{n+1}(\mathbf{R}^n \setminus \{0\}).$$

*To prove this, we decompose the above Fourier multipliers into high frequency part and low frequency part as in the proof of Theorem 2.1 below.*

*As for high frequency part, an application of Hörmander-Mihlin multiplier theorem for the multiplier $\widehat{\phi}(\xi/|\xi|)$, $\widehat{\psi}(\xi/|\xi|)$ reduces the problem to the radially symmetric case, where necessary calculations are carried out explicitly and everything is smooth. On the regularity of $\widehat{\phi}$, $\widehat{\psi}$, see Lemma 2.1. As for low frequency part, the Hardy-Littlewood-Sobolev inequarity enables us to obtain the desired result.*

We see that Theorem 2.2 follows from Theorem 2.1. Thus, we devote this section below to the proof of Theorem 2.1. We first collect the elementary lemmas.

**Lemma 2.1** *Let $0 < p < n$ and let $k \in \mathbf{N}$ satisfy $p-n+k \geq 1$. If $f \in C^k(\mathbf{R}^n \setminus \{0\})$ is homogeneous of degree $-p$, then $\widehat{f}$, Fourier transform of $f$ in $\mathcal{S}'(\mathbf{R}^n)$, is homogeneous of degree $-n+p$ and*

$$\widehat{f} \in C^{[p-n+k]-}(\mathbf{R}^n \setminus \{0\}),$$

*where $\mathcal{S}'$ is the space of tempered distributions.*

*Proof.* It is well known that $\widehat{f}$ is homogenious of degree $-n+p$. So we show the regularity of $\widehat{f}$ away from the origin. We set $\rho$ be a smooth cut of function such that $\rho(x) = 1$ if $|x| \leq 1$, $\rho(x) = 0$ if $|x| \geq 2$, and divide $\widehat{f}$ using $\rho$:

$$\widehat{f} = (\rho f)\widehat{\ } + \big((1-\rho)f\big)\widehat{\ }.$$

It is clear that $(\rho f)\widehat{}\in C^\infty(\mathbf{R}^n)$ and therefore we consider the regularity of $((1-\rho)f)\widehat{}$.

We first consider the case $p \notin \mathbf{N}$. For multi-indices $\alpha$, $\beta$ with

$$|\alpha| \leq [p-n+k], \quad |\beta| = k,$$

we have

$$\xi^\alpha \partial^\alpha((1-\rho)f)\widehat{}= C_{\alpha,\beta}\{(1-\rho)\partial^\beta(x^\alpha f)\} + g_{\alpha,\beta},$$

for a suitable constant $C_{\alpha,\beta}$ and $g_{\alpha,\beta} \in \mathcal{S}(\mathbf{R}^n)$, the space of smooth functions of rapid decrease. Then, we observe that $\partial^\beta(x^\alpha f)$ is a homogeneous function of degree $-p+|\alpha|-k$. This implies that $(1-\rho)\partial^\beta(x^\alpha f) \in L^1(\mathbf{R}^n)$, since

$$-p+|\alpha|-k \leq -p-k+[p-n+k] < -n.$$

Therefore, we conclude that $\xi^\beta\partial^\alpha((1-\rho)f)\widehat{}$ is identified with a continuous function as long as $|\alpha| \leq [p-n+k]$ and thus $\widehat{f} \in C^{[p-n+k]}(\mathbf{R}^n \setminus \{0\})$.

In the case of $p \in \mathbf{N}$, we conclude that $\widehat{f} \in C^{p-n+k-1}(\mathbf{R}^n \setminus \{0\})$ in the same manner as above. The regularity is determined by the range of $\alpha$ that is to be $|\alpha| \leq p-n+k-1$ so that $-p+|\alpha|-k \leq -n-1$. □

Now we define the dilation operator $D_{\lambda,p}$ by

$$D_{\lambda,p}u(t,x) = \lambda^p u(\lambda t, \lambda x), \quad t>0,\ x \in \mathbf{R}^n$$

for $\lambda$, $p > 0$.

**Lemma 2.2** *Let $n \geq 2$ and let $0 < p < n-1$. We assume that $\phi$, $\psi \in \mathcal{S}'$ are homogeneous of degree $-p$, $-p-1$, respectively. Then we have in $\mathcal{S}'(\mathbf{R}^n)$,*

$$D_{\lambda,p}\cos[(-\Delta)^{\frac{1}{2}}t]\phi = \cos[(-\Delta)^{\frac{1}{2}}t]\phi$$
$$D_{\lambda,p}(-\Delta)^{-\frac{1}{2}}\sin[(-\Delta)^{\frac{1}{2}}t]\psi = (-\Delta)^{-\frac{1}{2}}\sin[(-\Delta)^{\frac{1}{2}}t]\psi$$

*for each $t > 0$.*

**Remark 2.3** *In particular if we take $\lambda = 1/t$, then we obtain*

$$\cos[(-\Delta)^{\frac{1}{2}}t]\phi(x) = t^{-p}\cos[(-\Delta)^{\frac{1}{2}}]\phi(x/t),$$
$$(-\Delta)^{-\frac{1}{2}}\sin[(-\Delta)^{\frac{1}{2}}t]\psi(x) = t^{-p}(-\Delta)^{-\frac{1}{2}}\sin[(-\Delta)^{\frac{1}{2}}t]\psi(x/t).$$

*Proof.* This lemma follows by the relation with dilation operators and Fourier transform, together with the homogeneity of $\phi$, $\psi$. We only prove the latter, since the former follows similarly. We first notice that

$$D_{\lambda,p}(-\Delta)^{-\frac{1}{2}}\sin[(-\Delta)^{\frac{1}{2}}t]\psi = \lambda^p D_\lambda \mathcal{F}^{-1}\left[|\xi|^{-1}\sin\lambda t|\xi|\,\widehat{\psi}(\xi)\right], \tag{2.1}$$

where $D_\lambda f(x) = f(\lambda x)$. The right hand side of (2.1) equals to

$$\lambda^p \mathcal{F}^{-1}\big[\lambda^{-n} D_{1/\lambda}\big(|\xi|^{-1}\sin\lambda t|\xi|\,\widehat{\psi}(\xi)\big)\big],$$

by the relation with dilation operators and Fourier transform. This completes the proof, since $\widehat{\psi}$ is homogeneous of degree $-n+p+1$. □

**Lemma 2.3** *Let* $0<\alpha<1$. *If* $(1+|x|)^\alpha f \in L^1(\mathbf{R}^n)$, *then we have* $\widehat{f} \in C^{0,\alpha}(\mathbf{R}^n)$.

*Proof.* Since $|e^{-ix\cdot\xi} - e^{-ix\cdot\eta}| \le \min\{|x||\xi-\eta|, 2\}$, we obtain

$$\begin{aligned}
\frac{|\widehat{f}(\xi) - \widehat{f}(\eta)|}{|\xi-\eta|^\alpha} &\le (2\pi)^{-\frac{n}{2}} \int \frac{|e^{-ix\cdot\xi} - e^{-ix\cdot\eta}|}{|\xi-\eta|^\alpha}|f(x)|dx \\
&\le (2\pi)^{-\frac{n}{2}} \int \frac{\min\{|x||\xi-\eta|, 2\}}{|x|^\alpha|\xi-\eta|^\alpha}(1+|x|)^\alpha|f(x)|dx \\
&\le (2\pi)^{-\frac{n}{2}} 2^{1-\alpha}\|(1+|x|)^\alpha f\|_{L^1}. \qquad \square
\end{aligned}$$

Now we consider the behavior of the Fourier transform of functions associated with solutions of *FW*. Its precise form is the following. Let $a \in C^\infty(\mathbf{R}^n \setminus \{0\})$ be homogeneous of degree $-\lambda$. We set

$$\begin{aligned}
K_\varepsilon^\pm(x) &= \mathcal{F}^{-1}\big[e^{-\varepsilon|\xi|}\eta(\xi)a(\xi)e^{\pm i|\xi|}\big](x), \\
K_s(x) &= \mathcal{F}^{-1}\big[\rho(\xi)a(\xi)\sin|\xi|\big](x), \\
K_c(x) &= \mathcal{F}^{-1}[\rho(\xi)a(\xi)\cos|\xi|](x),
\end{aligned}$$

where $\eta \in C^\infty(\mathbf{R}^n)$ is a radial function satisfying $\eta(\xi) = 0$ if $|\xi| \le 1$, $\eta(\xi) = 1$ if $|\xi| \ge 2$, and $\rho = 1-\eta$. We notice that $K_0^\pm = \lim_{\varepsilon\downarrow 0} K_\varepsilon^\pm$ correspond to a high frequency part of solutions to $FW$ and $K_s$, $K_c$ a low frequency part.

**Proposition 2.1** *Let* $\lambda > 0$ *and let* $K_0^+ = \lim_{\varepsilon\downarrow0} K_\varepsilon^+$. *Then* $K_0^+ \in C^\infty(\mathbf{R}^n \setminus S^{n-1})$ *and the following holds.*

(1) *For* $0 < \lambda < \frac{n+1}{2}$,

$$K_0^+(x) = A_\lambda^+ a(-x/|x|)(1-|x|+i0)^{\lambda-\frac{n+1}{2}} + o(|1-|x||^{\lambda-\frac{n+1}{2}}) \tag{2.2}$$

*as* $|x| \to 1$, *where* $A_\lambda^+ = (2\pi)^{-\frac{1}{2}} e^{\frac{\pi}{2}i(-\lambda+n)}\Gamma(-\lambda+\frac{n+1}{2})$.

(2) *For* $\lambda = \frac{n+1}{2}$,

$$K_0^+(x) = -A^+ a(-x/|x|)\log(1-|x|+i0) + O(1), \tag{2.3}$$

*as* $|x| \to 1$, *where* $A^+ = (2\pi)^{-\frac{1}{2}} e^{\frac{(n-1)\pi}{4}i}$.

(3) *For* $\lambda > \frac{n+1}{2}$, $\varepsilon > 0$,

$$K_0^+ \in C^{[\lambda-\frac{n+1}{2}]_-, \lambda-\frac{n+1}{2}-[\lambda-\frac{n+1}{2}]_- -\varepsilon}(\mathbf{R}^n).$$

(4) *For* $N \geq 1$, $K_0^+ = o(|x|^{-N})$ *as* $|x| \to \infty$.

**Remark 2.4** *A similar result holds for* $K_0^- = \lim_{\varepsilon \downarrow 0} K_\varepsilon^-$. *In particular, we have*

$$K_0^-(x) = \begin{cases} A_\lambda^- a(x/|x|)(1-|x|-i0)^{\lambda-\frac{n+1}{2}} + o(\left|1-|x|\right|^{\lambda-\frac{n+1}{2}}) & \text{if } 0 < \lambda < \frac{n+1}{2}, \\ -A^- a(x/|x|)\log(1-|x|-i0) + O(1) & \text{if } \lambda = \frac{n-1}{2}, \end{cases}$$

*as* $|x| \to 1$, *where* $A_\lambda^- = (2\pi)^{-\frac{1}{2}} e^{-\frac{\pi}{2}i(-\lambda+n)}\Gamma(-\lambda+\frac{n+1}{2})$, $A^- = (2\pi)^{-\frac{1}{2}} e^{\frac{(n-1)\pi}{4}i}$.

For the proof of this proposition we refer to Miyachi [5](Proposition 2). The Hölder continuity follows from Lemma 2.3.

**Proposition 2.2** *For* $0 < \lambda < n$, *we have* $K_s \in C^\infty(\mathbf{R}^n)$ *and*

$$K_s = O(|x|^{\lambda-n-1}) \quad as \quad |x| \to \infty. \tag{2.4}$$

**Remark 2.5** *For* $K_c$, *we have*

$$K_c = O(|x|^{\lambda-n}) \quad as \quad |x| \to \infty. \tag{2.5}$$

*The difference of asymptotic behavior with* $K_s$ *comes from that of the behavior between* $\sin|\xi|$ *and* $\cos|\xi|$ *near* $|\xi| = 0$.

*Proof.* The regularity of $K_s$ follows from the fact that $K_s$ is the inverse Fourier transform of an integrable function with compact support. Thus, we prove the asymptotic behavior of $K_s$ in the proposition.

Representing the integral by polar coordinates, we have

$$\begin{aligned} K_s(x) &= (2\pi)^{-\frac{n}{2}} \int_0^2 \rho(s) s^{n-\lambda-1} \sin s \Big( \int_{S^{n-1}} e^{isr\omega\cdot\theta} a(\theta) d\sigma(\theta) \Big) ds \\ &= (2\pi)^{-\frac{n}{2}} r^{\lambda-n-1} \int_0^{2r} \rho(s/r) s^{n-\lambda-1} r \sin(s/r) \Big( \int_{S^{n-1}} e^{is\omega\cdot\theta} a(\theta) d\sigma(\theta) \Big) ds \\ &= (2\pi)^{-\frac{n}{2}} r^{\lambda-n-1} I_\omega(r), \end{aligned}$$

where we set $x = r\omega$ with $r > 0$, $\omega \in S^{n-1}$, and denote by $d\sigma$ the surface measure on $S^{n-1}$. Thus, it is sufficient to show

$$\sup_\omega \sup_{r>1} |I_\omega(r)| < \infty. \tag{2.6}$$

For the proof of (2.6), the behavior of the oscillatory integral

$$\int_{S^{n-1}} e^{is\omega\cdot\theta}a(\theta)d\sigma(\theta) \tag{2.7}$$

as $s \to \infty$ plays an important role. For that purpose we use the stationary phase method. Since the stationary points of the phase function $\omega \cdot \theta$ are $\pm\omega$, we divide (2.7) using partition of unity $\{\varphi,\ \varphi^+ \varphi^-\}$ and then divide $I_\omega$ as follows.

$$\begin{aligned} I_\omega(r) &= \int_0^{2r} e^{is}\rho(s/r)s^{n-\lambda-1}r\sin(s/r)\Big(\int_{S^{n-1}} e^{is(\omega\cdot\theta-1)}\varphi^+(\theta)a(\theta)d\sigma(\theta)\Big)ds \\ &\quad+\int_0^{2r} e^{-is}\rho(s/r)s^{n-\lambda-1}r\sin(s/r)\Big(\int_{S^{n-1}} e^{is(\omega\cdot\theta+1)}\varphi^-(\theta)a(\theta)d\sigma(\theta)\Big)ds \\ &\quad+\int_0^{2r} e^{is}\rho(s/r)s^{n-\lambda-1}r\sin(s/r)\Big(\int_{S^{n-1}} e^{is\omega\cdot\theta}\varphi(\theta)a(\theta)d\sigma(\theta)\Big)ds \\ &\equiv I_\omega^+(r) + I_\omega^-(r) + R_\omega(r), \end{aligned}$$

where $\varphi^+ + \varphi^- + \varphi \equiv 1$ and $\varphi^+$, $\varphi^-$ are supported in small neighborhoods of $\omega$, $-\omega$, respectively.

We see that $\sup_\omega \sup_{r>1} |R_\omega(r)| < \infty$, since the corresponding oscillatory integral over the unit sphere has a rapid decrease in $s$. More precisely,

$$\Big|\int_{S^{n-1}} e^{is\omega\cdot\theta}\varphi(\theta)a(\theta)d\sigma(\theta)\Big| \le C_N(1+s)^{-N}, \quad s > 0,$$

for all $N \ge 1$, which follows from the fact that the support of $\varphi a$ does not contain the stationary points of the phase function.

To prove that $\sup_\omega \sup_{r>1} |I_\omega(r)| < \infty$, we first notice that (see Stein [10], Chap. VIII)

$$\begin{aligned} \Big|\Big(\frac{d}{ds}\Big)^k A_\omega^+\Big| &= \Big|\int_{S^{n-1}} e^{is(\omega\cdot\theta-1)}e^{\frac{k\pi}{2}i}(\omega\cdot\theta-1)^k\varphi^+a(\theta)d\sigma(\theta)\Big| \\ &\le C(1+s)^{-\frac{n-1}{2}-k}, \quad s>0,\ k\in \mathbf{N}\cup\{0\}, \end{aligned}$$

where we set

$$A_\omega^+(r) = \int_{S^{n-1}} e^{is(\omega\cdot\theta-1)}\varphi^+(\theta)a(\theta)d\sigma(\theta).$$

Then, we have

$$|I_\omega^+(r)| \le C\int_0^1 s^{n-\lambda}ds + \Big|\int_1^{2r} e^{is}\rho(s/r)s^{n-\lambda-1}r\sin(s/r)\,A_\omega^+(s)ds\Big|. \tag{2.8}$$

To estimate the second term on the right hand side of (2.8) we use integration by parts several times up to obtain enough decay on $s$. Indeed, we have

$$\Big|\Big(\frac{d}{ds}\Big)^k\{\rho(s/r)s^{n-\lambda-1}r\sin(s/r)\,A_\omega^+(s)\}\Big| \le Cs^{-\lambda+\frac{n+1}{2}-k}, \quad 0 < s < 2r,$$

for $k \in \mathbf{N} \cup \{0\}$. Therefore we obtain (2.6).

Finally, $\sup_\omega \sup_{r>1} |I_\omega^-(r)| < \infty$ is proved in the same way as above. This completes the proof. □

**Proof of Theorem 2.1** We only prove the estimates on $(-\Delta)^{-\frac{1}{2}} \sin[(-\Delta)^{\frac{1}{2}}t]\psi$ here, since the estimates on $\cos[(-\Delta)^{\frac{1}{2}}t]\phi$ follow similarly.

By Lemma 2.2, it is sufficient to show the theorem in the case $t = 1$. In fact,

$$\begin{aligned} &t(t+|x|)^{-\frac{n+1}{2}} \big|t-|x|\big|^{-p+\frac{n-1}{2}} (-\Delta)^{-\frac{1}{2}} \sin[(-\Delta)^{\frac{1}{2}}t]\psi(x) \\ &= (1+|x/t|)^{-\frac{n+1}{2}} \big|1-|x/t|\big|^{-p+\frac{n-1}{2}} (-\Delta)^{-\frac{1}{2}} \sin[(-\Delta)^{\frac{1}{2}}]\psi(x/t), \end{aligned}$$

if $\frac{n-1}{2} < p < \frac{n+1}{2}$, for example. To prove the estimates we use cut off functions $\rho$, $\eta$ which are defined before and decompose $(-\Delta)^{-\frac{1}{2}} \sin[(-\Delta)^{\frac{1}{2}}]\psi$ into high frequency part and low frequency part.

Since $\widehat{\psi}$ is smooth away from the origin and homogeneous of degree $p-n+1$ by Lemma 2.1, we have

$$\begin{aligned} (-\Delta)^{-\frac{1}{2}} \sin[(-\Delta)^{\frac{1}{2}}]\psi =& \mathcal{F}^{-1}\left[|\xi|^{-1} \sin|\xi|\, \widehat{\psi}(\xi)\right] \\ =& \mathcal{F}^{-1}\left[|\xi|^{p-n} \widehat{\psi}(\xi/|\xi|) \sin|\xi|\right] \\ =& \lim_{\varepsilon \downarrow 0} \mathcal{F}^{-1}\left[e^{-\varepsilon|\xi|}\eta(\xi)|\xi|^{p-n}\widehat{\psi}(\xi/|\xi|)\sin|\xi|\right] \\ &+ \mathcal{F}^{-1}\left[\rho(\xi)|\xi|^{p-n}\widehat{\psi}(\xi/|\xi|)\sin|\xi|\right] \\ =& \frac{1}{2i}(K_0^+ - K_0^-) + K_s, \end{aligned}$$

where we follow the same notation as in the preceding propositions and we regard $a(\xi)$ as $|\xi|^{p-n}\widehat{\psi}(\xi/|\xi|)$.

The above limit is taken in the sense of tempered distributions. By the condition $p < \frac{n+1}{2}$, we observe that the limiting function is locally integrable and that the above caluculation is justified.

Therefore the above function is bounded by constant multiple of

$$\begin{cases} (1+|x|)^{-\frac{n+1}{2}} \big|1-|x|\big|^{-p+\frac{n-1}{2}}, & \text{if } \frac{n-1}{2} < p < \frac{n+1}{2}, \\ (1+|x|)^{-\frac{n+1}{2}} \left(1+\log^-\big|1-|x|\big|\right), & \text{if } p = \frac{n-1}{2}, \\ (1+|x|)^{-p-1}, & \text{if } 0 < p < \frac{n-1}{2}, \end{cases}$$

by propositions 2.1 and 2.2. The required regularity also follows from those propositions. □

# 3 Optimality

In this section we consider optimality of the theorems in the preceding section. In particular, it is shown that we cannot expect better results if the initial data are radially symmetric.

Throughout this section, we fix

$$\phi(x) = C_1|x|^{-p}, \quad \psi(x) = C_2|x|^{-p-1}$$

for $C_1$, $C_2 \in \mathbf{R} \setminus \{0\}$ and $p > 0$. We first mention that Theorem 2.1 is optimal in the following sense.

**Theorem 3.1** *Let $n \geq 2$. Then the following holds for each $t > 0$.*

(1) *For $\frac{n-1}{2} < p < \frac{n+1}{2}$, we have*

$$\lim_{|x| \to t+0} \left|t^2 - |x|^2\right|^{p-\frac{n-1}{2}} \left|\cos[(-\Delta)^{\frac{1}{2}}t]\phi(x)\right| > 0, \quad \textit{if} \ \ (n-p)/2 \notin \mathbf{N},$$

$$\lim_{|x| \to t-0} \left|t^2 - |x|^2\right|^{p-\frac{n-1}{2}} \left|\cos[(-\Delta)^{\frac{1}{2}}t]\phi(x)\right| > 0, \quad \textit{if} \ \ (p+1)/2 \notin \mathbf{N},$$

$$\lim_{|x| \to t+0} \left|t^2 - |x|^2\right|^{p-\frac{n-1}{2}} \left|(-\Delta)^{-\frac{1}{2}} \sin[(-\Delta)^{\frac{1}{2}}t]\psi(x)\right| > 0,$$

$$\textit{if} \ \ (n-p-1)/2 \notin \mathbf{N},$$

$$\lim_{|x| \to t-0} \left|t^2 - |x|^2\right|^{p-\frac{n-1}{2}} \left|(-\Delta)^{-\frac{1}{2}} \sin[(-\Delta)^{\frac{1}{2}}t]\psi(x)\right| > 0, \quad \textit{if} \ \ p/2 \notin \mathbf{N}.$$

(2) *For $p = \frac{n-1}{2}$, we have*

$$\lim_{|x| \to t} \left(\log^- |1 - |x/t||\right)^{-1} \left|\cos[(-\Delta)^{\frac{1}{2}}t]\phi(x)\right| > 0, \quad \textit{if} \ \ (p+1)/2 \notin \mathbf{N},$$

$$\lim_{|x| \to t} \left(\log^- |1 - |x/t||\right)^{-1} \left|(-\Delta)^{-\frac{1}{2}} \sin[(-\Delta)^{\frac{1}{2}}t]\psi(x)\right| > 0, \quad \textit{if} \ \ p/2 \notin \mathbf{N}.$$

(3) *For $0 < p < \frac{n+1}{2}$, we have*

$$\lim_{|x| \to \infty} |x|^p \left|\cos[(-\Delta)^{\frac{1}{2}}t]\phi(x)\right| > 0,$$

$$\lim_{|x| \to \infty} |x|^{p+1} \left|(-\Delta)^{-\frac{1}{2}} \sin[(-\Delta)^{\frac{1}{2}}t]\psi(x)\right| > 0.$$

From Theorem 3.1 we obtain the following theorem which implies the intervals on $1/r$ in Theorem 2.2 is almost best possible.

**Theorem 3.2** *Let $n \geq 2$ and $t > 0$.*

(1) *If $\cos[(-\Delta)^{\frac{1}{2}}t]\phi \in L^r(\mathbf{R}^n)$ for some $p$, $r$ with $0 < p < n-1$, $1 \leq r \leq \infty$, then*

$$\begin{cases} p - \frac{n-1}{2} < \frac{1}{r} < \frac{p}{n}, & \textit{when} \ \frac{n-1}{2} < p < n-1, \\ 0 \leq \frac{1}{r} < \frac{p}{n}, & \textit{when} \ 0 < p < \frac{n-1}{2}. \end{cases}$$

(2) *If* $(-\Delta)^{\frac{1}{2}}\sin[(-\Delta)^{\frac{1}{2}}t]\psi \in L^r(\mathbf{R}^n)$ *for some* $p$, $r$ *with* $0 < p < n-1$, $1 \le r \le \infty$, *then*

$$\begin{cases} p - \frac{n-1}{2} < \frac{1}{r} < \frac{p+1}{n}, & \text{when } \frac{n-1}{2} < p < n-1, \\ 0 \le \frac{1}{r} < \frac{p+1}{n}, & \text{when } 0 < p < \frac{n-1}{2}. \end{cases}$$

**Remark 3.1** *(1) By the first conclusions on* $1/r$*, we see that* $p$ *must be less than* $n/2$ *in both cases.*
*(2) In the case of* $p = (n-1)/2$*, it depends on the dimension* $n$ *whether the interval on* $1/r$ *contains* $0$ *or not. See Theorem 3.1(2).*
*(3) In the case of* $\frac{n-1}{2} < p < \frac{n}{2}$ *and* $1 < r < \infty$ *in Theorem 3.2(2), we can remove the condition on* $\psi$ *as follows:* $\psi$ *is homogeneous of degree* $-p-1$ *and*

$$\psi \in C^{n+1}(\mathbf{R}^n \setminus \{0\}), \quad \widehat{\psi}(\omega) \neq 0 \ \text{ for } \ \omega \in S^{n-1}.$$

*Under the above conditions, we can reduce radially symmetric case, since* $(-\Delta)^{-\frac{1}{2}}\sin[(-\Delta)^{\frac{1}{2}}t]\psi \in L^r(\mathbf{R}^n)$ *is equivalent to*

$$\mathcal{F}^{-1}\left[|\xi|^{p-n}\sin t|\xi|\right] \in L^r(\mathbf{R}^n)$$

*by Hörmander-Mihlin multiplier theorem. In the above argument, the fact* $|\xi|^{p-n}\sin t|\xi| \in L^2(\mathbf{R}^n)$ *is important.*

To prove Theorem 3.1 we prepare a lemma below. Before stating the lemma, we introduce some notation.

When $\mu$, $\nu \in \mathbf{C}$ satisfy $-\Re\nu - 1/2 < \Re\mu < 1$, then the integral

$$\int_0^\infty s^{\mu-1/2}J_\nu(s)ds,$$

converges as an improper integral, where $J_\nu$ is the Bessel function of order $\nu$, and its value is equal to

$$J(\nu, \mu) \equiv 2^{\mu-1/2}\Gamma\Big(\frac{\mu+\nu}{2}+\frac{1}{4}\Big)\Big/\Gamma\Big(\frac{\nu-\mu}{2}+\frac{3}{4}\Big).$$

**Lemma 3.1** *Let* $\mu$, $\nu \in \mathbf{R}$ *satisfy* $\mu + \nu > \frac{1}{2}$*. Then we have*

$$\lim_{R\to\infty}\int_0^\infty J_\nu(s)s^{\mu-1/2}\tau(s/R)ds$$
$$= \prod_{j=0}^{[\mu_+]-1}(-\mu+\nu+3/2+2j)J(\nu+[\mu_+], \mu-[\mu_+]),$$

*where* $\tau \in C_0^\infty(\mathbf{R})$ *with* $\tau(0) = 1$ *and* $\mu_+ = \max(\mu, 0)$*, with the convention that* $\prod_{j=0}^{-1}(-\mu+\nu+3/2+2j) = 1$.

*Proof.* We use the recursive formula for the Bessel functions

$$\frac{d}{dz}(z^{\nu+1}J_{\nu+1}(z)) = z^{\nu+1}J_\nu(z), \tag{3.1}$$

as well as the asymptotic formula

$$J_\nu(s) = \begin{cases} O(s^\nu) & \text{as } s \to 0, \\ O(s^{-1/2}) & \text{as } s \to \infty. \end{cases}$$

First we consider the case $\mu < 1$. Since

$$\int_0^\infty J_\nu(s)s^{\mu-1/2}\tau(s/R)ds = J(\nu,\mu) - \int_0^\infty J_\nu(s)s^{\mu-1/2}(1-\tau(s/R))ds,$$

it suffices to show that the second term on the right hand side converges to zero as $R \to \infty$. Using (3.1) and integrating by parts we have

$$\begin{aligned} &\int_0^\infty J_\nu(s)s^{\mu-1/2}(1-\tau(s/R))ds \\ &= (-\mu+\nu+3/2)\int_0^\infty J_{\nu+1}(s)s^{\mu-3/2}(1-\tau(s/R))ds \\ &\quad + \int_0^\infty J_{\nu+1}(s)s^{\mu-1/2}\tau'(s/R)R^{-1}ds, \end{aligned} \tag{3.2}$$

where the boundary values vanish, since

$$J_{\nu+1}(s)s^{\mu-1/2}(1-\tau(s/R)) = \begin{cases} O(s^{\mu+\nu+1/2}) & \text{as } s \to 0, \\ O(s^{\mu-1}) & \text{as } s \to \infty. \end{cases}$$

Then the first term on the right hand side of (3.2) converges to zero as $R \to \infty$ by Lebesgue's dominated convergence theorem.

Meanwhile, the second term on the right hand side of (3.2) is estimated by

$$CR^{-1}\Big(\int_0^1 s^{\mu+\nu+\frac{1}{2}}ds + \int_1^{CR} s^{\mu-1}ds\Big) \le CR^{\mu-1},$$

which converges to zero as $R \to \infty$. This completes the proof for the case $\mu < 1$.

In what follows we consider the case $\mu \geq 1$. In this case we use the following identity

$$J_\nu(s) = s^{-\nu}\Big(s^{-1}\frac{d}{ds}\Big)^k (s^{\nu+k}J_{\nu+k}(s))$$

for $k \in \mathbf{N} \cup \{0\}$. This follows by a succesive use of the recursive formula (3.1). Using this and integrating by parts, we obtain, with suitable constants $C_l$,

$$\begin{aligned}
&\int_0^\infty J_\nu(s) s^{\mu-1/2} \tau(s/R)\, ds \\
&= \int_0^\infty \Big\{\Big(s^{-1}\frac{d}{ds}\Big)^{[\mu]} (s^{\nu+[\mu]} J_{\nu+[\mu]}(s))\Big\} s^{\mu-\nu-1/2} \tau(s/R)\, ds \\
&= \prod_{j=0}^{[\mu]-1} (-\mu+\nu+3/2+2j) \int_0^\infty s^{\nu+[\mu]} J_{\nu+[\mu]}(s) s^{\mu-\nu-1/2-2[\mu]} \tau(s/R)\, ds \\
&\quad + \sum_{l=1}^{[\mu]} C_l \int_0^\infty s^{\nu+[\mu]} J_{\nu+[\mu]}(s) s^{\mu-\nu-1/2-2[\mu]+l}\, \tau^{(l)}(s/R) R^{-l}\, ds, \qquad (3.3)
\end{aligned}$$

where the boundary values at $s = 0$ also vanish, since

$$\begin{aligned}
&\Big\{\Big(s^{-1}\frac{d}{ds}\Big)^{[\mu]-m} (s^{\nu+[\mu]} J_{\nu+[\mu]}(s))\Big\} s^{\mu-\nu-1/2-m-(m-l-1)}\, \tau^{(l)}(s/R) R^{-l} \\
&= O(s^{\mu+\nu+1/2+l}) \quad \text{as } s \to 0,
\end{aligned}$$

for $0 \le m \le [\mu]-1,\ 0 \le l \le m-1$.

As for the first term on the right hand side of the last equality of (3.3) we apply the preceding argument since $\mu - [\mu] < 1,\ (\mu - [\mu]) + (\nu + [\mu]) > -1/2$. So the first term on the right hand side of the last equality of (3.3) converges to

$$\prod_{j=0}^{[\mu]-1} (-\mu+\nu+3/2+2j) J(\nu+[\mu], \mu-[\mu])$$

as $R \to \infty$.

Meanwhile, integrating by parts once more, the other terms on the right hand side of the last equality of (3.3) are estimated by

$$\sum_{l=1}^{[\mu]+1} C_l' R^{-l} \Big( \int_0^1 s^{\nu+\mu-1/2+l} ds + \int_1^{CR} s^{\mu-[\mu]+l-2} ds \Big) \le C R^{\mu-[\mu]-1},$$

which converges to zero as $R \to \infty$. This completes the proof. □

**Proof of Theorem 3.1** We only prove the results on $(-\Delta)^{-\frac{1}{2}} \sin[(-\Delta)^{\frac{1}{2}} t]\psi$ here with $t = 1$, as in the proof of Theorem 2.1. From the proof of Theorem 2.1, we observe that

$$(-\Delta)^{-\frac{1}{2}} \sin[(-\Delta)^{\frac{1}{2}} t]\psi = \frac{1}{2i}(K_0^+ - K_0^-) + K_s.$$

Here we regard $a(\xi)$ as $|\xi|^{-1}\widehat{\psi}(\xi) = C_{n,p}|\xi|^{p-n}$, where

$$C_{n,p} = C_2\, 2^{n/2-p-1} \Gamma\Big(\frac{n-p-1}{2}\Big) \Big/ \Gamma\Big(\frac{p+1}{2}\Big).$$

We first prove (1), (2). Applying Proposition 2.1, we obtain

$$\lim_{|x|\to 1-0} \left|1-|x|\right|^{p-\frac{n-1}{2}} (-\Delta)^{-\frac{1}{2}} \sin[(-\Delta)^{\frac{1}{2}} t]\psi(x) \tag{3.4}$$

$$= \frac{1}{2i}(A^+_{n-p} - A^-_{n-p})\, C_{n,p}\,, \tag{3.5}$$

$$\lim_{|x|\to 1+0} \left|1-|x|\right|^{p-\frac{n-1}{2}} (-\Delta)^{-\frac{1}{2}} \sin[(-\Delta)^{\frac{1}{2}} t]\psi(x)$$

$$= \frac{1}{2i}\{e^{(-p+(n-1)/2)\pi i} A^+_{n-p} - e^{(p-(n-1)/2)\pi i} A^-_{n-p}\}\, C_{n,p}\,, \tag{3.6}$$

if $\frac{n-1}{2} < p < \frac{n+1}{2}$, since $K_s$ is bounded and

$$(1-|x| \pm i0)^{-\lambda} = (|x|-1)^{-\lambda} e^{\mp\lambda\pi i},$$

when $\lambda \in \mathbf{R}$, $|x| > 1$. Then, (3.5) is equal to

$$(2\pi)^{-1/2}\, C_{n,p}\, \Gamma\Big(p - \frac{n-1}{2}\Big) \sin\frac{p\pi}{2}\,,$$

and (3.6) is equal to

$$(2\pi)^{-1/2}\, C_{n,p}\, \Gamma\Big(p - \frac{n-1}{2}\Big) \sin\frac{(n-p-1)\pi}{2}\,.$$

Therefore, we obtain

$$\lim_{|x|\to 1\pm 0} \left|1-|x|\right|^{p-\frac{n-1}{2}} (-\Delta)^{-\frac{1}{2}} \sin[(-\Delta)^{\frac{1}{2}} t]\psi(x) \neq 0,$$

except for the case $(n-p-1)/2 \in \mathbf{N}$, $p/2 \in \mathbf{N}$, respectively.

As for the case $p = \frac{n-1}{2}$, we obtain from Proposition 2.1

$$\lim_{|x|\to 1} \big(\log^- \left|1-|x|\right|\big)(-\Delta)^{-\frac{1}{2}} \sin[(-\Delta)^{\frac{1}{2}} t]\psi(x) = (2\pi)^{-1/2}\, C_{n,p} \sin\frac{p\pi}{2}\,,$$

from which we obtain the desired result.

On the other hand, to prove (3) we need to consider the asymptotic behavior of $K_s$, since $K_0^{\pm}$ decrease rapidly by Proposition 2.1. Using the representation formula of the Fourier transform for radially symmetric functions, we have

$$\begin{aligned} K_s(x) &= C_{n,p}\, \mathcal{F}^{-1}\big[\rho(\xi)|\xi|^{p-n} \sin|\xi|\big](x) \\ &= C_{n,p}\, |x|^{-p-1} \int_0^\infty J_{n/2-1}(s) s^{p-n/2+1} \tau(s/|x|) ds, \end{aligned}$$

where $\tau(r) = (\rho(r)/r)\sin r$. We notice that $\tau \in C_0^\infty(\mathbf{R})$ and $\tau(0) = 1$. Thus, it suffices to show that

$$\lim_{|x|\to\infty} \int_0^\infty J_{n/2-1}(s) s^{p-n/2+1} \tau(s/|x|) ds \neq 0. \tag{3.7}$$

To show (3.7) we apply Lemma 3.1, since

$$\left(p - \frac{n-3}{2}\right) + \left(\frac{n}{2} - 1\right) = p + \frac{1}{2} > \frac{1}{2}.$$

Therefore, the left hand side of (3.7) is equal to

$$\prod_{j=0}^{m-1} (n - p - 1 + 2j)\, J\left(\frac{n}{2} - 1 + m,\, p - \frac{n-3}{2} - m\right),$$

and its value is not equal to 0 for $0 < p < \frac{n+1}{2}$, where we set $m = [(p - (n-3)/2)_+]$. This completes the proof. □

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